UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH2101**
ASSESSMENT PATTERN	:	**MATH2101A**
MODULE NAME	:	**Analysis 3: Complex Analysis**
DATE	:	**01-May-08**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2007/08-MATH2101A-001-EXAM-228

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) (i) Give the definition of the derivative of a function f at a point $z_0 \in \mathbb{C}$.
   (ii) What does it mean for a function f to be holomorphic in the domain $\Omega \subset \mathbb{C}$?

   (b) Assuming that the function f is holomorphic in the disk $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ prove that $g(z) = \overline{f(\overline{z})}$ is also holomorphic in $D(0,1)$ and find its derivative.

   (c) Find the radii of convergence of the following series, stating clearly which results you are using:

$$\sum_{k=0}^{\infty} k^{113} 2^{-k} (z-1)^k, \quad \sum_{n=2}^{\infty} n!(z-e)^{3n}, \quad \sum_{k=0}^{\infty} \frac{z^k}{(k!)^2}.$$

2. (a) State the Cauchy-Riemann equations for real-valued functions $u(x,y), v(x,y)$.

   (b) Suppose that $f(z) = u(x,y) + iv(x,y)$ is holomorphic on a domain D, and that $|f(z)| = \text{const}$ throughout D. Show that $f(z) = \text{const}$ as well.

   (c) (i) Write the MacLaurin series for $\sin z$ and $\cos z$.
   (ii) Using the sum formula for the cosine function, write the Taylor expansion of $\cos z$ centered at z_0 (i.e. expansion in powers of $z - z_0$), where z_0 is a point in $\mathbb{C}$.
   (iii) Prove that $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$, stating clearly which results you are using.
   (iv) Prove that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

3. (a) Suppose that the functions $f(z) = u(x, y)+iv(x, y)$ and $g(z) = v(x, y)+iu(x, y)$ are analytic in some domain D. Show that both u and v are constant functions.

   (b) Let f be a holomorphic function on the punctured disk $D'(0, R) = \{z \in \mathbb{C} : 0 < |z| < R\}$, where $R > 0$ is fixed. Write down the formulae for c_n in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

   Using these formulae, prove that if f is bounded on $D'(0, R)$, it has a removable singularity at 0. [10]

   (c) Find the maximal radius $R > 0$ for which the function $f(z) = (\sin z)^{-1}$ is holomorphic in $D'(0, R)$, and find the principal part of its Laurent expansion about $z_0 = 0$.

4. (a) State the Cauchy integral formula.

   (b) Let $\Gamma_1 = \{z \in \mathbb{C} : |z - 3i| = 2\}$ and $\Gamma_2 = \{z \in \mathbb{C} : |z| = 5\}$. be two positively oriented circular contours. Evaluate the integrals

$$I_1 = \int_{\Gamma_1} \frac{z}{z^2+4} dz, \qquad I_2 = \int_{\Gamma_2} \frac{z}{z^2+4} dz,$$

   clearly stating which results you are using.

   (c) Using the Cauchy-Riemann equations or otherwise prove that the function $h(z) = \sin(\mathrm{Im} z)$ is not differentiable at any point of the strip $\{z : -\pi/2 < \mathrm{Im} z < \pi/2\}$.

5. (a) Describe three types of isolated singularities of a function f by explaining how they are related to the principal part of its Laurent expansion.

   (b) Find the Laurent expansions of the function

$$f(z) = \frac{z}{z^2 - 1}$$

   valid for (i) $0 < |z - 1| < 2$, (ii) $|z + 1| > 2$, (iii) $|z| > 1$.

6. (a) Give the definition of the residue $\mathrm{Res}(f, z_0)$ of a function f at the point z_0.

   (b) Find $\mathrm{Res}(g, 0)$ for $g(z) = z^{-2} \cosh z$.

   (c) Evaluate the following integral by integrating around a suitable closed contour:

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2 + 4} dx.$$